FRATTINI SUBGROUPS

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by

John Malcolm CAMPBELL

B.Sc.(Hons), University of Bristol, England.

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PREFACE

I thank Dr. M.F. Newman for supervising the thesis, and for his many valuable criticisms and suggestions. The main source of inspiration has been a lecture course on Frattini subgroups which was given in 1963 by Dr. Newman at the Australian National University.

I also thank Dr. L.G. Kovács for his helpful advice while Dr. Newman was in the United States.

INTRODUCTION

The Frattini subgroup, $\varphi(G)$, of a group G is the intersection of all the maximal subgroups of G, or else G itself if G has no maximal subgroups. In 1885 Giovanni Frattini showed that $\varphi(G)$ is nilpotent if G is finite. Since then, many results have been published on Frattini subgroups, and a reasonably complete Bibliography is given after the list of references.

In this thesis, the author has attempted to survey the more important material on Frattini subgroups, and has included a number of results of his own. Inevitably, some important material has been omitted because of the present bulk of the literature.

The organisation and content of the thesis are indicated by the chapter and section headings.

Chapter I sets out the definitions and notation which are used in the thesis.

Apart from sections 4.3.3 - 4.3.7 the material in Chapters II - IV is based on Dr. Newman's lecture notes on Frattini subgroups. In Chapter II, the author presents B.H. Neumann's well-known characterisation of the Frattini subgroup in terms of non-generators, and gives three sufficient conditions for the Frattini subgroup of a group to be omissible. The question of omissibility of the entire Frattini subgroup is returned to in Chapter VI and again in Chapter VII. In Chapter III the author gives a sufficient condition for a subset of a group to be omissible, and discusses the effect of homomorphisms on omissible subsets. The chapter closes with two sources of omissible normal subgroups, the first due to D.H. McLain [15], and the second due to R.L. Taylor (unpublished).

In Chapter IV, the central theme is the Frattini subgroup in relation to other subgroups. The author proves that for any group G,

$$\beta_*(G) \cap \zeta_*(G) \leq \varphi(G) \leq \beta_*(G),$$

and that $\varphi(G) = \beta_*(G)$ if and only if every maximal subgroup of G is normal. Because $\varphi = \beta_*$ for such a wide class of groups, the author collects some properties of β_* in section 4.3. The chapter closes with a proof that if G is any group, and if K is the union of all the normal periodic nilpotent subgroups of G, then $\varphi(G)$ contains $\beta_*(K)$.

Chapter V describes two simple results concerning the behaviour of the Frattini subgroup under homomorphisms. The first appears as a lemma in B.H. Neumann [20], and the second is a generalisation of the first.

Chapter VI is concerned with abelian groups, and begins with a characterisation of non-generators in terms of divisibility. The remainder of the chapter is devoted to omissible subsets. The author proves that a subset S of an abelian group G is omissible in G if and only if

(a) $S \leq \varphi(G)$

(b) the subgroup generated by S has no non-trivial divisible factor group.

Condition (b) leads the author to examine D-groups (abelian groups having no non-trivial divisible factor group). The chapter closes with a characterisation of aperiodic D-groups of rank 1.

In Chapter VII the author returns to the question of omissibility of the Frattini subgroup. He defines a B-group as a group which can be extended to a finitely generated group such that the factor group is polycyclic, and proves that $\varphi(F)$ is omissible in F if F is a nilpotent B-group. The author was led to consider nilpotent B-groups when attempting to study the Frattini subgroups of finitely generated metanilpotent groups.

Chapter VIII deals with finite groups having a normal nilpotent Hall subgroup. The author shows that if H is a normal nilpotent Hall π-subgroup of a finite group G, then $\varphi(H)$ is the Hall π-subgroup of $\varphi(G)$, and

$$\varphi(G) = T \times \varphi(H),$$

where T is the Hall π'-subgroup of $\varphi(G)$. As a corollary, if the Fitting radical $\mathfrak{F}(G)$ is a Hall subgroup of G, then

$$\varphi(G) = \varphi(\mathfrak{F}(G)).$$

He also proves that if H is a normal nilpotent Hall subgroup of a finite soluble group G, then for every complement S of H in G,

$$\varphi(G) = (\varphi(G) \cap \varphi(S)) \times \varphi(H).$$

In Chapter IX, the author turns to L1-groups (a finite group is an L1-group if it has p-length at most 1 for every prime p), and proves the following theorem:

If G is an L1-group, and for each prime p, G_p is a Sylow p-subgroup of G, then $\varphi(G)$ is the direct product of the subgroups D_p, where

$$D_p = \bigcap_{g \in G} \varphi(G_p)^g.$$

This extends a result obtained by B. Huppert [13] for finite groups having a nilpotent commutator subgroup. The chapter closes with an example, due to Dr. L.G. Kovács, of a finite group G having 3-length 2 and p-length ≤ 1 $(p \neq 3)$ for which $\bigcap_p D_p < \varphi(G)$.

In Chapter X, the author examines the problem of determining which groups occur as Frattini subgroups, and begins by proving that each abelian group H has an abelian extension K such that

$$H = \varphi(K)$$

and that there is, in a certain sense, a minimal choice for K (the author was unaware at the time that V. Dlab [2] had already proved this result). The author then shows that an omissible normal finite subgroup of a group is nilpotent (the proof is based on Dr. Newman's lecture notes), and concludes the chapter with an example of a finite nilpotent group (the Dihedral group of order 8) which cannot be extended to a group in which it is omissible.

Chapters Xl and Xll are based almost entirely on Dr. Newman's lecture notes. Chapter Xl is mainly devoted to proving the following theorem due to M.F. Newman:

If a group G contains normal subgroups D and H such that

(a) $D \leq H \cap \varphi(G)$,

(b) H/D is nilpotent,

(c) D is polycyclic-by-finite,

then H is nilpotent.

The chapter closes with a simple application of Newman's theorem.

Chapter XII discusses an open question on direct products, and is devoted to proving the following result due to V. Dlab and V. Kořínek [3]:

There exist groups A and B such that $\varphi(A \times B) \neq \varphi(A) \times \varphi(B)$ if and only if there is a simple group without maximal subgroups.

STATEMENT

The material reported in this thesis is the author's own work except where references to other sources are explicitly stated in the Introduction and text.

John M. Campbell

John M. Campbell.

CHAPTER I

DEFINITIONS AND NOTATION

1.1 THE FRATTINI SUBGROUP

The Frattini subgroup (after G. Frattini) of a group G, written $\varphi(G)$, is defined as the intersection of all the maximal subgroups of G, or else G itself if G has no maximal subgroup.

1.2 CENTRAL SERIES, THE HYPERCENTRE AND COMMUTATOR SUBGROUPS

The <u>upper central series</u> of a group G is an ascending series of characteristic subgroups $\zeta_\alpha(G)$ which are defined as follows:

$\zeta_o(G) = E$ (the unit subgroup)

$\zeta_{\alpha+1}(G)/\zeta_\alpha(G)$ is the centre of $G/\zeta_\alpha(G)$

$\zeta_\beta(G) = \bigcup_{\alpha<\beta} \zeta_\alpha(G)$ if β is a limit ordinal.

The subgroup at which the upper central series becomes stationary is termed the <u>hypercentre</u> of G, and is written $\zeta_*(G)$.

If x and y are elements of a group, the element $x^{-1}y^{-1}xy$, normally written $[x,y]$, is called their <u>commutator</u>.

Similarly, the <u>commutator</u> $[X,Y]$ of two subgroups X,Y of a group is defined as the subgroup generated by all commutators $[x,y]$ where x,y range over X,Y respectively.

Higher (left normed) commutators of elements and subgroups are defined recursively by the rules

$[x_1,x_2,\ldots,x_n] = [[x_1,x_2,\ldots,x_{n-1}],x_n]$ for elements

$[X_1,X_2,\ldots,X_n] = [[X_1,X_2,\ldots,X_{n-1}],X_n]$ for subgroups.

The <u>lower central series</u> of a group G is a descending series of fully invariant subgroups $\gamma_\alpha(G)$ which are defined as follows:

$\gamma_1(G) = G$

$\gamma_{\alpha+1}(G) = [\gamma_\alpha(G), G]$

$\gamma_\beta(G) = \bigcap_{\alpha<\beta} \gamma_\alpha(G)$ if β is a limit ordinal.

$\gamma_2(G)$, the commutator subgroup of G, is alternatively written $\delta(G)$.

1.3 THE SUBGROUPS $\beta_p(G)$, $\beta_*(G)$

If p is a prime and G a group, $\beta_p(G)$ is defined as the subgroup of G generated by all the p-th powers of elements of G. Its importance stems from the fact that for abelian p-groups, $\beta_p(G)$ coincides with $\varphi(G)$.

If G is any group, $\beta_*(G)$ is defined as the intersection of the subgroups $\beta_p(G).\delta(G)$, where p ranges over all primes. For a wide class of groups, $\beta_*(G)$ coincides with $\varphi(G)$.

1.4 OMISSIBLE SUBSETS AND NON-GENERATORS

A subset S of a group G is said to be <u>omissible</u> in G if whenever S and any subset T together generate G, then T alone generates G.

An element x of G is called a <u>non-generator</u> if the subset comprising x alone is omissible in G.

For example, if G is a quasicyclic p-group, then every proper subgroup of G is omissible in G, and every element of G is a non-generator.

The concept of a non-generator is important because it leads to an alternative characterization of the Frattini subgroup.

1.5 THE FITTING RADICAL

The <u>Fitting radical</u> $\mathfrak{F}(G)$ of a group G is defined as the subgroup generated by all the normal nilpotent subgroups of G.

H. Fitting [6] has shown that the subgroup generated by two normal nilpotent subgroups of a finite group is itself nilpotent. The same is true for infinite groups (see, for example, M. Hall [8]). Consequently, $\mathfrak{F}(G)$ is always locally nilpotent, and will be the unique largest normal nilpotent subgroup of G if G satisfies the maximal condition on normal nilpotent subgroups.

W. Gaschütz [7] showed that for a finite group G, the natural homomorphism $G \longrightarrow G/\varphi(G)$ maps the Fitting radical of G on to the Fitting radical of $G/\varphi(G)$.

P. Hall [12] then showed that this remains valid if G is any finite extension of a finitely generated metanilpotent group.

1.6 SYLOW SUBGROUPS AND HALL SUBGROUPS.

If π is a set of prime numbers, the complementary set of primes is denoted by π'. A positive integer is called a <u>π-number</u> if all its prime divisors belong to π.

If G is a group, a periodic subgroup H is called a *π-subgroup* of G if the order of every element of H is a π-number. A maximal π-subgroup of G is called a *Sylow π-subgroup* of G.

A subgroup H of a finite group G is called a *Hall subgroup* of G if its order and index (in G) are coprime. More specifically, H is a Hall *π-subgroup* of G if $|H|$ is a π-number and $[G:H]$ is a π'-number.

CHAPTER II

OMISSIBILITY AND THE FRATTINI SUBGROUP

2.1 A CHARACTERISATION OF THE FRATTINI SUBGROUP IN TERMS OF NON-GENERATORS

B.H. Neumann [16] showed that the Frattini subgroup of a group coincides with the set of all non-generators of the group. To prove this, let G be any group, and suppose that an element x lies outside $\varphi(G)$. If M is a maximal subgroup of G not containing x, then x and M together generate G yet M alone does not. Consequently, x is not a non-generator of G. Conversely, suppose that x is not a non-generator, and let T be a subset of G such that x and T together generate G but T alone does not. Let M be a subgroup of G which is maximal with respect to containing T but not x (some form of the Axiom of Choice is needed to guarantee the existence of M). It is easy to see that M is a maximal subgroup of G. Since x does not lie in M, it cannot lie in $\varphi(G)$.

2.2 SOME SUFFICIENT CONDITIONS FOR THE FRATTINI SUBGROUP TO BE OMISSIBLE

2.2.1 If $\varphi(G)$ is finitely generated, then $\varphi(G)$ is omissible in G.

To prove this, let $\{x_1, x_2, \ldots, x_k\}$ be a set of generators of $\varphi(G)$, and suppose that $\varphi(G)$ and T together generate G. Then $x_1, x_2, \ldots, x_k$ and T together generate G. Since the elements x_i are, by 2.1, non-generators, each one may be dropped in turn from the generating set $\{x_1, x_2, \ldots, x_k\} \cup T$, showing that T alone generates G.

2.2.2 If every proper subgroup of G is contained in a maximal subgroup of G, then $\varphi(G)$ is omissible in G. (H.J. Zassenhaus [17]).

For suppose that T generates a proper subgroup of G, and let M be a maximal subgroup of G containing T. Then $\varphi(G)$ and T are both contained in M so cannot together generate G.

2.2.3 If a group G is finitely generated then $\varphi(G)$ is omissible in G.

It is well known that finitely generated groups satisfy the hypothesis of 2.2.2 so that 2.2.3 follows from 2.2.2. We also give an independent proof: Suppose that $\{x_1, x_2, \ldots, x_n\}$ is a set of generators of G, and that $\varphi(G)$ and T together generate G. Express each x_i as a word in elements of $\varphi(G)$ and T, and let S comprise all the elements of $\varphi(G)$ which appear in these words. Then S and T together generate G. But S may be dropped because it is a finite set of non-generators, showing that T generates G.

CHAPTER III

OMISSIBLE SUBSETS

Because of the close connection between omissible subsets and the Frattini subgroup, we present below several facts about omissible subsets, some of which are used in later chapters.

3.1 A SUFFICIENT CONDITION FOR OMISSIBILITY

The following result is often useful when looking for omissible subsets:

3.1.1 If S is a normal subset of a group G, and if S is omissible in some subgroup H of G, then S is omissible in G itself.

<u>Proof</u> Suppose that S and T together generate G, and let N,K be the subgroups generated by S,T respectively. Then N is normal in G, so that

$$G = NK,$$

and so $H = N(H \cap K)$.

Since S is omissible in H, this reduces to

$$H = H \cap K,$$

in other words, K contains H. Therefore K contains N, giving

$$G = NK = K$$

i.e., T generates G.

3.2 THE EFFECT OF HOMOMORPHISMS ON OMISSIBLE SUBSETS

There are two fairly simple results of interest in this direction:

3.2.1 If θ is a homomorphism of a group G into some group $\bar{G}$, and if S is omissible in G, then $S\theta$ is omissible in $G\theta$.

Proof Suppose that $S\theta$ and T together generate $G\theta$, and let Y be the complete inverse image of T under θ. Then S and Y together generate G, so that Y alone does by the omissibility of S in G. Consequently, $T = Y\theta$ generates $G\theta$.

The converse of 3.2.1 fails in the sense that it is not always possible to "lift" an omissible subset of $G\theta$ to an omissible subset of G. For example, let G be an infinite cyclic group, and θ a homomorphism which maps G on to a cyclic group H of order 4. Then H contains an omissible subgroup of order 2 having no omissible inverse image in G. However, we have the following partial converse to 3.2.1.

3.2.2 Let θ be a homomorphism of a group G into some group $\bar{G}$, and let K be the kernel of θ. Suppose that K is omissible in G, and that S^* is omissible in $G\theta$. Then every inverse image of S^* is omissible in G.

Proof Let S be an inverse image of S^*, and suppose that S and T together generate G. Then $S\theta = S^*$ and $T\theta$ together generate $G\theta$, hence $T\theta$ alone since S^* is omissible in $G\theta$. Consequently, G is generated by T and K, therefore by T alone since K is omissible in G.

3.3 TWO SOURCES OF OMISSIBLE NORMAL SUBGROUPS

If G is any group, then every omissible subset of G, and in particular every omissible normal subgroup of G, lies in $\varphi(G)$, although $\varphi(G)$ itself may not be omissible. There are two sources of omissible normal subgroups, the first due to D.H. McLain [15], and the second due to R.L. Taylor (unpublished), which we now state as theorems.

3.3.1 If N is a normal subgroup of a group G, then $\delta(N) \cap \zeta_n(N)$ is an omissible normal subgroup of G for all positive integers n.

<u>Proof</u> Let $N_n = \delta(N) \cap \zeta_n(N)$.

Clearly N_n is characteristic in N, therefore normal in G.
We use induction on n to prove the omissibility of N_n.
N_o is the unit subgroup, which is omissible.
Suppose that N_{n-1} is omissible $(n \geq 1)$, and that N_n and T together generate G. Let T generate the subgroup K. Then $G = KN_n$.
If we can show that $G = KN_{n-1}$, then we shall have $G = K$ from the omissibility of N_{n-1}, and it will follow that N_n is omissible.
It therefore remains to show that $G = KN_{n-1}$.
We have

$$\delta(N) = [N,N] = [KN_n \cap N,N] = [(K \cap N)N_n,N]$$

$$\leq [K \cap N,N][N_n,N]$$

$$\leq [K \cap N,N]N_{n-1}$$

and $[K \cap N, N] = [K \cap N, KN_n \cap N] = [K \cap N, (K \cap N)N_n]$

$$\leq [K \cap N, K \cap N][K \cap N, N_n]$$

$$\leq KN_{n-1}, \text{ so that } \delta(N) \leq KN_{n-1}.$$

Therefore

$$G = KN_n \leq K.\delta(N) \leq KN_{n-1}, \text{ i.e., } G = KN_{n-1} \text{ as required.}$$

3.3.1 has the following important corollary for nilpotent groups:

<u>Corollary</u> If G is nilpotent, then $\delta(G)$ is omissible in G.

For if G has class c, then $\zeta_c(G) = G$, giving $\delta(G) \cap \zeta_c(G) = \delta(G)$.

3.3.2 If S is an omissible subset of G, and if its normal closure N is nilpotent, then N is omissible in G.

<u>Proof</u> Suppose that N and T together generate G, and let T generate a subgroup H. Then $G = HN$, and we must show that $G = H$. The essential step in the proof is to show that H and S together generate G, and then apply the omissibility of S. We have

$$N = \text{sgp.}\{s^g : s \in S, g \in G\}$$

$$= \text{sgp.}\{s^{hx} : s \in S, h \in H, x \in N\}$$

$$= \text{sgp.}\{s^h : s \in S, h \in H\}.\delta(N) \text{ since } s^{hx} = s^h[s^h, x].$$

But $\delta(N)$ is omissible in N because N is nilpotent, so that

$$N = \mathrm{sgp.}\left\{s^h : s \in S,\ h \in H\right\}$$

Therefore

$$G = HN = \mathrm{sgp.}\left\{H \cup S\right\} \text{ as required.}$$

CHAPTER IV

THE FRATTINI SUBGROUP IN RELATION TO OTHER SUBGROUPS

In this chapter, we attempt to relate the Frattini subgroup to various verbal or verbally defined subgroups.

4.1 SUBGROUPS CONTAINED IN THE FRATTINI SUBGROUP

We begin by proving

4.1.1 For any group G, $\varphi(G)$ contains $\beta_*(G) \cap \zeta_*(G)$

Proof Firstly, we require the following lemma regarding the hypercentre of a group:

Lemma If a subgroup H of a group G does not contain $\zeta_*(G)$, then H is properly contained in its normaliser.
To see this, let α be the first ordinal such that H does not contain $\zeta_\alpha(G)$. Then $\alpha = \beta+1$, where H contains $\zeta_\beta(G)$. Therefore

$$[\zeta_{\beta+1}(G),H] \leq [\zeta_{\beta+1}(G),G] \leq \zeta_\beta(G) \leq H$$

Consequently, $\zeta_{\beta+1}(G)$ lies in the normaliser of H, but not in H itself.

To complete the proof of 4.1.1, let M be a maximal subgroup of G. If M is normal in G, then G/M is cyclic of order p where p is prime. Hence $\beta_p(G).\delta(G) \leq M$, and so $\beta_*(G) \leq M$.
On the other hand, if M is not normal in G, then M is self-normalising, and by the lemma must contain $\zeta_*(G)$.
In either case, M contains $\beta_*(G) \cap \zeta_*(G)$, proving 4.1.1 .

4.2 THE CLASS OF GROUPS FOR WHICH $\varphi(G) = \beta_*(G)$

Firstly, we show that for any group G, $\beta_*(G)$ contains $\varphi(G)$. To see this, observe that for each prime p, $G/\beta_p(G).\delta(G)$ is an elementary abelian p-group and therefore has a trivial Frattini subgroup. Consequently, $\beta_p(G).\delta(G)$ is an intersection of certain maximal subgroups of G (precisely those which are normal and of index p), so that $\beta_p(G).\delta(G)$ contains $\varphi(G)$. This is true for each prime p, giving $\beta_*(G) \geq \varphi(G)$.

The following theorem gives a characterization of the class of groups G for which $\varphi(G) = \beta_*(G)$. This class is very wide, and certainly contains all locally nilpotent groups.

4.2.1 If G is any group, then $\varphi(G) = \beta_*(G)$ if and only if every maximal subgroup of G is normal.

<u>Proof</u> We already have $\varphi(G) \leq \beta_*(G)$

If M is maximal and normal in G, then M contains $\beta_*(G)$ as in the proof of 4.1.1. Consequently, if every maximal subgroup is normal, $\varphi(G)$ contains $\beta_*(G)$.

Conversely, if $\varphi(G) = \beta_*(G)$, then every maximal subgroup of G contains the commutator subgroup $\delta(G)$ and is therefore normal in G.

4.3 SOME PROPERTIES OF $\beta_*(G)$

Since $\varphi(G) = \beta_*(G)$ for such a wide class of groups, it seems worth exploring some of the properties of $\beta_*(G)$.

4.3.1 β_* is <u>monotonic</u> in the sense that if H is a subgroup of a group G, then $\beta_*(H)$ is contained in $\beta_*(G)$.

To see this, observe that for each prime p, $\beta_p(H).\delta(H) \leq \beta_p(G).\delta(G)$.

In general, φ is not monotonic, even when the subgroup in question is normal. As an example, due to Dr. M.F. Newman, let G be the holomorph of the additive rationals by the multiplicative rationals: G is the set of all pairs (s,r) ($s \neq o$, r rational numbers) with group operation

$$(s_1,r_1)(s_2,r_2) = (s_1 s_2, r_1 s_2 + r_2)$$

The subgroup H of all pairs $(1,r)$ is isomorphic to the additive rationals. Consequently $\varphi(H) = H$.

On the other hand, $\varphi(G) = E$; in fact, the sets

$$B_1 = \left\{(s,\overset{0}{\cancel{1}}) : s \neq o,\ s \text{ rational}\right\}$$

$$B_2 = \left\{(s,1-s) : s \neq o,\ s \text{ rational}\right\}$$

are easily seen to be maximal subgroups of G which intersect trivially.

4.3.2 If G is the direct [cartesian] product of certain subgroups G_α, then $\beta_*(G)$ is [is contained in] the direct [cartesian] product of the subgroups $\beta_*(G_\alpha)$.

This is a simple consequence of the definition of β_*.

4.3.3 If G is any group, then the factor group $G/\beta_*(G)$ is isomprphic to a subgroup of the cartesian product $\bar{\prod_p}\{G/\beta_p(G).\delta(G)\}$

<u>Proof</u> Let $A = \bar{\prod_p}\{G/\beta_p(G).\delta(G)\}$, and define a homomorphism θ from G to A by the rule

$$(g\theta)_p = g.\beta_p(G).\delta(G)$$

The kernel of θ is seen to be precisely $\beta_*(G)$, thereby proving 4.3.3.

We observe in passing that the periodic subgroup of A is $\prod_p\{G/\beta_p(G).\delta(G)\}$ which is a direct product of elementary abelian groups. The same is therefore true of the periodic subgroup of $G/\beta_*(G)$, a fact which will be needed later.

4.3.4 If G is any group, then $\beta_*(G) = E$ if and only if G is isomorphic to a subgroup of a cartesian product of elementary abelian p-groups A_p, one for each prime p.

<u>Proof</u> If $\beta_*(G) = E$, then 4.3.3 shows that G is isomorphic to a subgroup of $\bar{\prod_p}\{G/\beta_p(G).\delta(G)\}$, so we need only take A_p equal to $G/\beta_p(G).\delta(G)$.

Conversely, if G is a subgroup of $\bar{\prod_p} A_p$ where each A_p is an elementary abelian p-group, then $\beta_*(G) \leq \beta_*(\bar{\prod_p} A_p) \leq \bar{\prod_p}(\beta_* A_p) = E$.

4.3.5 If a group G contains a normal periodic nilpotent subgroup H, then $\varphi(G)$ contains $\varphi(H)$.

The case when H is an abelian p-group has been given by P. Hall [12]. The group constructed in 4.3.1 shows that the assumption that H is periodic cannot be omitted.

<u>Proof</u> Firstly, suppose that H is an abelian p-group, and let M be a maximal subgroup of G. If M contains H then M contains $\varphi(H)$. Otherwise

$$G = HM$$

and $H \cap M$ is normal in G. It follows from the maximality of M in G that $H/H \cap M$ is characteristically simple, therefore an elementary abelian p-group. Hence

$$\varphi(H) = \beta_p(H) \leq H \cap M$$

so that M contains $\varphi(H)$.
Consequently $\varphi(H) \leq \varphi(G)$.

Secondly, if H is any periodic abelian group, then

$$H = \prod_p H_p$$

where H_p is the Sylow p-subgroup of H, and the case just proved gives

$$\varphi(H_p) \leq \varphi(G).$$

But for abelian groups, $\varphi = \beta_*$ so that by 4.3.2,

$$\varphi(H) = \bigcap_p \varphi(H_p).$$

Hence $\varphi(H) \leq \varphi(G)$.

Thirdly, suppose that H is any periodic nilpotent group. By 3.3.1, $\delta(H) \leq \varphi(H)$ so that

$$\varphi(H/\delta(H)) = \varphi(H)/\delta(H).$$

By 3.3.1, $\delta(H)$ is omissible in H so that by 3.1.1, $\delta(H) \leq \varphi(G)$. Hence

$$\varphi(G/\delta(H)) = \varphi(G)/\delta(H).$$

Finally,

$$\varphi(H/\delta(H)) \leq \varphi(G/\delta(H))$$

by the case just proved, since $H/\delta(H)$ is a normal abelian periodic subgroup of $G/\delta(H)$. Hence $\varphi(H) \leq \varphi(G)$.

4.3.6 If G is a periodic group, then

$$\beta_*(G) = \bigcup_H \beta_*(H)$$

where H runs through all the finitely generated subgroups of G.

Proof 4.3.1 implies that $\bigcup_H \beta_*(H) \leq \beta_*(G)$. Conversely, let $x \in \beta_*(G)$, and let π be the set of prime divisors of the order of x. Since π is finite, G contains a finitely generated subgroup H such that

$$x \in \bigcap_{p \in \pi} \beta_p(H).\delta(H) .$$

If $p \in \pi'$, there exists an integer m such that $x = x^{mp}$. But then, $x \in \beta_p(H)$, and so

$$x \in \bigcap_{\text{all } p} \beta_p(H).\delta(H) = \beta_*(H),$$

proving 4.3.6.

4.3.7 If G is any group, and if K is the periodic part of its Fitting radical, then $\varphi(G)$ contains $\beta_*(K)$.

Proof K is clearly the union of all the normal periodic nilpotent subgroups of G. Let $x \in \beta_*(K)$. By 4.3.6, K contains a finitely generated subgroup H such that $x \in \beta_*(H)$. Since H is finitely generated, and the subgroup generated by finitely many normal periodic nilpotent subgroups of G is again such a subgroup, one such subgroup, say L, must contain H.

Then $\quad x \in \beta_*(L) = \varphi(L)$,

and $\quad \varphi(L) \leq \varphi(G)$ by 4.3.5.

Therefore $x \in \varphi(G)$, and 4.3.7 is proved.

CHAPTER V

THE BEHAVIOUR OF THE FRATTINI SUBGROUP UNDER HOMOMORPHISMS

If θ is a homomorphism of a group G into some group $\overline{G}$, then in general all we can say is that $(\varphi(G))\theta$ is contained in $\varphi(G\theta)$. This is because $G\theta$ has "fewer" maximal subgroups than G in the sense that the maximal subgroups of $G\theta$ come from those maximal subgroups of G which contain the kernel of θ. We now state and prove this formally.

5.1 Let θ be a homomorphism from a group G to some group $\overline{G}$, and let K be the kernel of θ. Then

$$\varphi(G\theta) \geq (\varphi(G))\theta$$

with equality if $\varphi(G)$ contains K. (B.H. Neumannn [20]).

<u>Proof</u> If M is a maximal subgroup of G, then $M\theta$ is maximal in $G\theta$ or equal to $G\theta$ according as M does or does not contain K, and all the maximal subgroups of $G\theta$ are obtained in this way. Consequently,

$$\varphi(G\theta) = H\theta$$

where H is the intersection of all the maximal subgroups of G which contain K. Therefore $H \geq \varphi(G)$, and $\varphi(G\theta) = H\theta \geq (\varphi(G))\theta$.

If $\varphi(G)$ contains K, then every maximal subgroup of G contains K, so that $H = \varphi(G)$, and $\varphi(G\theta) = H\theta = (\varphi(G))\theta$.

A similar result, which will be needed later, is the following.

5.2. If K is a normal subgroup of a group G, and if H is any subgroup of G, then

$$\varphi(KH/K) \geq K.\varphi(H)/K$$

with equality if $\varphi(H)$ contains $K \cap H$.

By taking H equal to G, and K equal to the kernel of θ, we see that 5.1 is a special case of 5.2.

<u>Proof of 5.2</u> We use the isomorphism $KL \longleftrightarrow L$ which exists between the lattice of subgroups KL between K and KH, and the lattice of subgroups L between $K \cap H$ and H. In fact, $\varphi(KH/K) = KL/K$ where L is the intersection of all the maximal subgroups of H which contain $K \cap H$. Therefore $L \geq \varphi(H)$, and $KL/K \geq K.\varphi(H)/K$. Furthermore, if $\varphi(H) \geq K \cap H$, then $L = \varphi(H)$ and we have equality.

CHAPTER VI

FRATTINI SUBGROUPS OF ABELIAN GROUPS

In this chapter we confine our attention to abelian groups, and "group" will always mean abelian group. Additive notation will normally be used. In chapter IV it was shown that $\varphi = \beta_*$ for a class of groups which included all abelian groups. Therefore all the properties of β_* apply in this context to φ.

6.1 A CHARACTERIZATION OF NON-GENETATORS IN TERMS OF DIVISIBILITY

If G is an abelian group, then

$$\varphi(G) = \beta_*(G) = \bigcap_p \beta_p(G)$$

where $\beta_p(G)$ now coincides with the set of p^{th} multiples of elements of G. Consequently, an element is a non-generator precisely when it is divisible (within G) by every prime. In particular, $\varphi(G) = G$ if and only if G is a divisible group.

At the other extreme, $\varphi(G)$ is trivial if and only if G is isomorphic to a subgroup of a complete direct sum of elementary abelian p-groups, one for each prime p. This is simply a re-statement of 4.3.4 in the abelian case.

6.2 OMISSIBLE SUBSETS

We pose the question: Which subsets of an abelian group G are omissible in G? The answer is provided by the following theorem.

6.2.1 A subset S of an abelian group G is omissible in G if and only if

(a) S is contained in $\varphi(G)$

(b) the subgroup generated by S has no non-trivial divisible factor group.

Proof Let $\{S\}$ denote the subgroup generated by S.
Firstly, suppose that $\{S\}$ is not omissible in G, and that $\{S\} \leq \varphi(G)$.
We must show that (b) fails.
G contains a proper subgroup H such that $H + \{S\} = G$
H cannot lie in a maximal subgroup M of G, for this would imply

$$H + \{S\} \leq H + \varphi(G) \leq M.$$

Hence the factor group G/H has no maximal subgroup and is therefore divisible. But

$$G/H = H + \{S\}/H \simeq \{S\}/H \cap \{S\}$$

so that $\{S\}$ has a non-trivial divisible factor group.

Conversely, suppose that (a) or (b) fails.

If (a) fails then clearly S is not omissible in G. But if (b) fails then $\{S\}$ has a non-trivial divisible factor group $\{S\}/K$, and by the splitting property of divisible groups, G contains a subgroup H such that

$$G/K = H/K \oplus \{S\}/K$$

Therefore

$$G = H \dotplus \{S\}.$$

On the other hand, H is a proper subgroup of G since

$$H \cap \{S\} = K < \{S\}.$$

Hence $\{S\}$ is not omissible in G. But then neither is S.

<u>Corollary</u> $\varphi(G)$ is omissible in G if and only if $\varphi(G)$ has no non-trivial divisible factor group.

6.3 D-GROUPS

The above criterion for omissibility is not altogether convenient because it is not quite clear which abelian groups have no non-trivial divisible factor group. We shall therefore attempt to classify or at least characterize such groups in terms of other group properties.

We begin with a definition.

An abelian group is a D-<u>group</u> if it has no non-trivial divisible factor group.

Clearly, a group is a D-group if and only if for each prime p it has no factor group of type $C(p^{\infty})$.

We give several elementary results regarding D-groups which will be used later.

6.3.1 Every subgroup and every factor group of a D-group is a D-group.

Proof The statement regarding factor groups is clear.

Regarding subgroups, suppose that a group G has a subgroup H having a non-trivial divisible factor group H/K. By the splitting property of divisible groups, G contains a subgroup L such that

$$G/K = H/K \oplus L/K.$$

But then G has the non-trivial divisible factor group G/L.

6.3.2 If H is a subgroup of a group G, then G is a D-group if and only if H and G/H are D-groups.

Proof If G is a D-group, then so are H and G/H by 6.3.1.

Conversely, suppose that G has a non-trivial divisible factor group G/L. If $L + H < G$, then G/H has the non-trivial divisible factor group

$$(G/H)/(L + H/H) \simeq G/L + H$$

But if $L + H = G$, then H has the non-trivial divisible factor group $H/L \cap H$.

Corollary 1. A direct sum of a finite number of D-groups is a D-group.

Corollary 2. A mixed group G is a D-group if and only if T and G/T are D-groups, where T is the maximal periodic subgroup of G.

In view of the last corollary, it is only necessary to explore the structure of periodic and aperiodic D-groups. We shall begin with periodic groups. The following theorem reduces the periodic to the primary case.

6.3.3 A periodic group G is a D-group if and only if for every prime p its Sylow p-subgroup is a D-group.

<u>Proof</u> Clearly, if G is a D-group, then so is every Sylow p-subgroup by 6.3.1. Conversely, if G has a non-trivial divisible factor group, then it has a factor group of type $C(p^\infty)$ for some prime p. But then so must its Sylow p-subgroup.

We now give the structure theorem for primary D-groups.

6.3.4 A p-group is a D-group if and only if it has finite exponent.

<u>Proof</u> Clearly, a p-group of finite exponent is a D-group. Conversely, suppose that a p-group G is a D-group, and let B be a basic subgroup of G. We cannot have $B < G$ for then G/B would be a non-trivial divisible factor group of G. Therefore G is a direct sum of cyclic groups. If G were not of finite exponent, it would have a subgroup H:

$$H = \bigoplus_{i=1}^{\infty} \{x_i\}$$

where x_i has order p^i.

But H would have the factor group H/K of type $C(p^{\infty})$, where K is the subgroup of H generated by the elements

$$y_i = p\, x_{i+1} - x_i \qquad (i = 1,2,\ldots)$$

However, this would contradict the assumption that G (and therefore every subgroup of G) is a D-group. Therefore G has finite exponent.

Combining 6.3.3 and 6.3.4, we obtain the following structure theorem for periodic D-groups.

6.3.5 A periodic group is a D-group if and only if for every prime p its Sylow p-subgroup has finite exponent.

Turning to aperiodic D-groups, we begin with free groups. We have

6.3.6 A free group is a D-group if and only if it has finite rank.

<u>Proof</u> Let F be a free group.

Firstly, suppose that F has finite rank. To show that F is a D-group it is enough, in view of corollary 1 to 6.3.2, to show that an infinite eyclic group is a D-group. But this is clear.

Secondly, suppose that F has infinite rank. Then F contains a subgroup H:

$$H = \bigoplus_{i=1}^{\infty} \{x_i\}$$

of countable rank.

H has the divisible factor group H/K (of rational type) where K is the subgroup of H generated by the elements

$$y_i = (i + 1)x_{i+1} - x_i \qquad (i = 1,2,\ldots)$$

Consequently H, and therefore F, is not a D-group.

We turn now to aperiodic D-groups in general. In view of 6.3.6 it is only necessary to consider groups of finite rank.

For each prime p, we shall need R_p, the ring of rational numbers whose denominators (in reduced form) are prime to p, and A_p, the underlying additive group of R_p.

The group A_p is an R_p-module by virtue of the multiplication in R_p. If G is any group, the module structure of A_p is inherited by the tensor product $A_p \otimes G$. Explicitly,

$$\pi(\sigma \otimes g) = (\pi\sigma) \otimes g \qquad (\pi \in R_p,\ \sigma \in A_p,\ g \in G).$$

We have the following theorem for aperiodic D-groups

6.3.7 An aperiodic group G of finite rank r is a D-group if and only if for every prime p, $A_p \otimes G$ is a direct sum of r copies of A_p.

Before proving 6.3.7, we list some elementary properties of modules and tensor products which will be needed for the proof. Proofs are outlined in brackets where necessary.

(a) Every exact sequence

$$0 \longrightarrow F \xrightarrow{\alpha} G \xrightarrow{\beta} H \longrightarrow 0$$

of groups and group-homomorphisms induces an exact sequence

$$0 \longrightarrow A_p \otimes F \xrightarrow{\alpha_*} A_p \otimes G \xrightarrow{\beta_*} A_p \otimes H \longrightarrow 0 \qquad \text{(E)}$$

of R_p-modules and R_p-homomorphisms.

α_* is related to α by

$$(\sigma \otimes g)\alpha_* = \sigma \otimes (g\alpha) \qquad (\sigma \in A_p,\ g \in G),$$

and similarly for β_* and β.

(The sequence (E) is exact since A_p is aperiodic: see, for example, S. MacLane [14]).

(b) If H is a periodic group, the group $A_p \otimes H$ is isomorphic to the Sylow p-subgroup of H.

(Express H as the direct sum $\sum_q H_q$ of its Sylow q-subgroups H_q. Then $A_p \otimes H \simeq \sum_q A_p \otimes H_q \simeq H_p$ since $A_p \otimes H_p \simeq H_p$, and $A_p \otimes H_q = 0$ if $q \neq p$).

(c) If M is an aperiodic R_p-module, the subgroup $p^k M$ is an R_p-submodule of M which is R_p-isomorphic to M.

(The map $x \longrightarrow p^k x$ $(x \in M)$ is an R_p-monomorphism).

(d) If N is an R_p-submodule of a free R_p-module M of finite rank r, then M has an R_p-base

$$u_1, u_2, \ldots, u_r$$

such that

$$p^{\ell_1} u_1, p^{\ell_2} u_2, \ldots, p^{\ell_s} u_s \qquad (s \leq r,\ 0 \leq \ell_1 \leq \ell_2 \leq \cdots \leq \ell_s)$$

is an R_p-base for N. In particular, N is free of rank $s \leq r$.

(This is proved in much the same way as the corresponding group theorem. The "torsion coefficients" are simpler in this case because p is the only prime element (apart from associates) in R_p.)

(e) If a free R_p-module M of finite rank r contains a submodule N of equal rank, then the group M/N is a direct sum of at most r cyclic p-groups.

(Put $s = r$ in (d) and examine M/N).

(f) If F is a free group of rank r, then $A_p \otimes F$ is R_p-free of R_p-rank r.

(If $F = \sum_{i=1}^{r} C_i$, each C_i infinite cyclic, then $A_p \otimes F \simeq \sum_{i=1}^{r} A_p \otimes C_i$ where each $A_p \otimes C_i \simeq A_p$).

(g) If G is aperiodic then so is $A_p \otimes G$.

In order to prove 6.3.7, let G be an aperiodic group of finite rank r, and let F be a free subgroup of G of equal rank.

The factor group G/F is periodic. Let $(G/F)_p$ be the Sylow p-subgroup of G/F.

By (a), the exact sequence

$$0 \longrightarrow F \longrightarrow G \longrightarrow G/F \longrightarrow 0$$

induces an exact sequence

$$0 \longrightarrow A_p \otimes F \longrightarrow A_p \otimes G \longrightarrow A_p \otimes (G/F) \longrightarrow 0$$

where, by (b), $A_p \otimes (G/F)$ is isomorphic to $(G/F)_p$.

We may therefore regard $A_p \otimes F$ as an R_p-submodule of $A_p \otimes G$ with quotient $(G/F)_p$.

$(G/F)_p$ is periodic so that $A_p \otimes F$ and $A_p \otimes G$ have the same R_p-rank. But $A_p \otimes F$ is R_p-free of R_p-rank r by (f). Therefore by (g), $A_p \otimes G$ is an aperiodic R_p-module of rank r. Hence to prove 6.3.7 it is enough to show that G is a D-group if and only if for every prime p, $A_p \otimes G$ is R_p-free.

Suppose firstly that G is a D-group. Then G/F is a D-group by 6.3.1 so that by 6.3.5 $(G/F)_p$ has finite exponent, say p^k.

Consequently

$$p^k(A_p \otimes G) \leq A_p \otimes F$$

so that by (d), $p^k(A_p \otimes G)$ is R_p-free. But $p^k(A_p \otimes G)$ is R_p-isomorphic to $A_p \otimes G$ by (c). Therefore $A_p \otimes G$ is R_p-free.

Conversely, suppose that $A_p \otimes G$ is R_p-free for every prime p. $A_p \otimes G$ contains the submodule $A_p \otimes F$ of equal rank so that by (e), the quotient $(G/F)_p$ is a finite p-group. This is true for every prime p so that by 6.3.5 G/F is a D-group.

F is in any case a D-group by 6.3.6 so that by 6.3.2 G itself is a D-group.

<u>Corollary</u> A rank 1 aperiodic group G of type $[m_1,m_2,\ldots]$ is a D-group if and only if every m_i is finite.

<u>Proof</u> Observe that $A_{p_i} \otimes G$ has type $[\infty,\ldots,\infty,m_i,\infty,\ldots]$ whereas A_{p_i} has type $[\infty,\ldots,\infty,0,\infty,\ldots]$ (0 in the i^{th} place).

Consequently, $A_{p_i} \otimes G \simeq A_{p_i}$ if and only if m_i is finite.

But G is a D-group if and only if $A_{p_i} \otimes G \simeq A_{p_i}$ for all i, and the corollary follows.

CHAPTER VII

FRATTINI SUBGROUPS OF NILPOTENT GROUPS

7.1 OMISSIBILITY OF THE FRATTINI SUBGROUP

In the corollary to 3.3.1 we showed that the commutator subgroup of a nilpotent group is omissible. In general the Frattini subgroup is not, but there is a close connection between nilpotent groups with omissible Frattini subgroup and abelian groups with the same property. We have:

7.1.1 If G is a nilpotent group, then $\varphi(G)$ is omissible in G if and only if $\varphi(G/\delta(G))$ is omissible in $G/\delta(G)$.

<u>Proof</u> Since $\delta(G) \leq \varphi(G)$, we have

$$\varphi(G/\delta(G)) = \varphi(G)/\delta(G).$$

But $\delta(G)$ is omissible in G so that by 3.2.1 and 3.2.2, $\varphi(G)$ is omissible in G if and only if $\varphi(G)/\delta(G)$ is omissible in $G/\delta(G)$.

7.2 THE B-GROUPS OF P. HALL

Following P. Hall [11] we shall call a group H a B-group if it can be extended to a finitely generated group G such that G/H is polycyclic. We prove the following theorem.

7.2.1 If F is a nilpotent B-group, then $\varphi(F)$ is omissible in F.

We require two lemmas, the first of which is due to P. Hall [11].

<u>Lemma 1</u> If A is an abelian B-group, then

$$A = T \times N$$

where T is a group of finite exponent, and N is an aperiodic group which contains a free abelian subgroup L such that N/L is a π-group for some finite set of primes π.

<u>Lemma 2</u> If G is an abelian group with a free (abelian) subgroup H such that G/H has a trivial Sylow p-subgroup for infinitely many primes p, then $H \cap \varphi(G) = E$.

<u>Proof of lemma 2</u>

Let x belong to $H \cap \varphi(G)$. Since $\varphi(G) = \bigcap_p \beta_p(G)$ for abelian groups there exists for each prime p an element x_p such that

$$x = x_p^p.$$

The assumption about G/H implies that infinitely many x_p lie in H. But the only element of H which is divisible in H by infinitely many primes is e. Therefore $x = e$.

<u>Proof of 7.2.1</u>

Let F be a normal nilpotent subgroup of a finitely generated group G with G/F polycyclic. By 7.1.1 it is enough to show that $\varphi(F/\delta(F))$ is omissible in $F/\delta(F)$. But $F/\delta(F)$ is an abelian B-group so that by lemma 1,

$$F/\delta(F) = T \times N .$$

where T has finite exponent and the aperiodic group N contains a free subgroup L such that N/L is a π-group for some finite set of primes π. By lemma 2,

$$L \cap \varphi(N) = E$$

so that $\varphi(N)$ is mapped faithfully into the periodic group N/L. Therefore

$$\varphi(N) = E.$$

Hence

$$\varphi(F/\delta(F)) = \varphi(T) \times \varphi(N) = \varphi(T).$$

But $\varphi(T)$ has finite exponent and is therefore (by 6.2.1) omissible in $F/\delta(F)$. Hence $\varphi(F/\delta(F))$ is omissible in $F/\delta(F)$ as required.

CHAPTER VIII

FRATTINI SUBGROUPS OF GROUPS HAVING A NORMAL HALL SUBGROUP

The author was led to consider such groups when studying the Frattini subgroups of finite metanilpotent groups. Some results obtained in this chapter are used in chapter IX on L1-groups.

We first observe that if H is a non-trivial normal Hall subgroup of a finite group G, then H cannot be omissible in G. For, by Schur's theorem, H is complemented in G by some subgroup K. If H were omissible, we should have

$$G = HK = K, \text{ giving } H = H \cap K = E.$$

This shows in particular that if $\varphi(G)$ is non-trivial, then $\varphi(G)$ cannot be a Hall subgroup of G. With a little more effort, we can prove the following stronger result for $\varphi(G)$.

3.1 If p is a prime divisor of the order of a finite group G, then p divides the index of $\varphi(G)$ in G.

<u>Proof</u> Suppose that p does not divide the index of $\varphi(G)$ in G, and let P be a Sylow p-subgroup of G. Then P is a Sylow p-subgroup of the nilpotent group $\varphi(G)$ and is therefore normal in G. P is omissible in G, and is obviously a Hall subgroup, so that by a previous observation, P is trivial. Therefore p does not divide the order of G either.

The remainder of this chapter is devoted to proving the following three theorems.

8.2 Let H be a normal nilpotent Hall subgroup of a finite group G. Let π be the set of primes dividing $|H|$, and π' the complementary set. Then

(a) $H \cap \varphi(G) = \varphi(H)$

(b) $\varphi(H)$ is the Sylow π-subgroup of $\varphi(G)$

(c) $\varphi(G) = T \times \varphi(H)$

where T is the Sylow π'-subgroup of $\varphi(G)$.

8.3 If the Fitting radical $\mathfrak{F}(G)$ is a Hall subgroup of the finite group G, then

$$\varphi(G) = \varphi(\mathfrak{F}(G)).$$

8.4 If H is a normal nilpotent Hall subgroup of a finite soluble group G, then for every complement S of H in G,

$$\varphi(G) = (\varphi(G) \cap \varphi(S)) \times \varphi(H)$$

Before proving these theorems, we shall prove a technical lemma.

8.5 Let H be a normal abelian subgroup of a group G such that

(a) H is a direct product of minimal G-normal subgroups K_α,

(b) H is complemented in G by a subgroup C.

Put $L_\beta = \prod_{\alpha \neq \beta} K_\alpha$. Then the subgroups $L_\beta C$ are maximal in G, and their intersection is C.

<u>Proof</u> $L_\beta C$ is a subgroup since L_β is normal in G.

$L_\beta C$ is a proper subgroup of G since $H \cap L_\beta C = L_\beta(H \cap C) = L_\beta < H$.

Now let x be an element of G not in $L_\beta C$.

Since $G = K_\beta L_\beta C$, we have

$$x = x_\beta y_\beta c \quad (e \neq x_\beta \in K_\beta,\ y_\beta \in L_\beta,\ c \in C).$$

Therefore $\text{sgp.}\{x, L_\beta C\} \geq \text{sgp.}\{x_\beta, L_\beta C\} \geq D.L_\beta C$, where $D = \text{sgp.}\{x_\beta^z : z \in C\}$ is normal in G and is contained in the minimal normal subgroup K_β. But $D \neq E$ so that $D = K_\beta$, giving

$$G = D.L_\beta C \leq \text{sgp.}\{x, L_\beta C\}.$$

This proves that $L_\beta C$ is maximal in G.

Now suppose that $x \in \bigcap_\beta (L_\beta C)$. x is uniquely expressible as a product $x = (\prod_\alpha x_\alpha).c \quad (x_\alpha \in K_\alpha,\ c \in C)$.

Since $x \in L_\beta C$, we have $x_\beta = e$. This is true for all β, giving

$$x = c \in C.$$

Therefore $\bigcap_\beta (L_\beta C) \leq C$.

The reverse inclusion is obvious.

<u>Proof of 8.2.(a)</u> We distinguish two cases

(α) H is a direct product of elementary abelian p-groups H_p, one for each prime p.

In this case, $\varphi(H) = E$, and we have to show that $H \cap \varphi(G) = E$ We begin by examining the representation which G induces in H_p. Specifically, let $\mathfrak{k}$ be the field of p elements and C be the centralizer of H_p in G. We turn H_p into a $\mathfrak{k}(G/C)$-module by setting

$$h(gC) = h^g \quad (h \in H_p, g \in G).$$

The [irreducible] submodules of H_p are precisely the [minimal] G-normal subgroups of H_p. Since C contains H, $[G:C]$ is prime to p so that Maschke's theorem applies (see [1]):

Every submodule of H_p has a complementary submodule.

Since H_p is finite, repeated application of Maschke's theorem leads to a direct product decomposition

$$H_p = K_1 \times K_2 \times \ldots \times K_t$$

where each K_i is minimal normal in G. By Schur's theorem, H_p is complemented in G by a subgroup S_p. We may now apply 8.5: S_p is an intersection of maximal subgroups of G.
But then, $H_p \cap \varphi(G) \le H_p \cap S_p = E$, that is, $\varphi(G)$ meets each H_p trivially. Therefore $H \cap \varphi(G) = E$.

(β) H is any normal nilpotent Hall subgroup of G.

Then $H/\varphi(H)$ is a normal nilpotent Hall subgroup of $G/\varphi(H)$. But for nilpotent groups, $\varphi = \beta_*$ so that by 4.3.3, $H/\varphi(H)$ is a direct product of elementary abelian groups. Therefore case (α) gives

$$H/\varphi(H) \cap \varphi(G/\varphi(H)) = E.$$

But $\varphi(G/\varphi(H)) = \varphi(G)/\varphi(H)$ since by 3.1.1 $\varphi(H) \leq \varphi(G)$. Therefore $H \cap \varphi(G) = \varphi(H)$, proving 8.2(a).

Proof of 8.2 (b) and (c).

The isomorphism $H.\varphi(G)/H \simeq \varphi(G)/H \cap \varphi(G)$ shows that $H \cap \varphi(G)$ is the Sylow π-subgroup of $\varphi(G)$. 8.2(b) now follows from 8.2(a). 8.2(c) follows from the fact that $\varphi(G)$ is a finite nilpotent group (see 10.2).

Proof of 8.3

Applying 8.2(a) with $H = \mathfrak{F}(G)$ we obtain

$$\mathfrak{F}(G) \cap \varphi(G) = \varphi(\mathfrak{F}(G)).$$

On the other hand, $\varphi(G)$ is nilpotent so that $\varphi(G) \leq \mathfrak{F}(G)$. Therefore $\varphi(G) = \varphi(\mathfrak{F}(G))$.

Proof of 8.4

Let G, H, S be defined as in 8.4. Then $H.\varphi(G)/H \leq \varphi(G/H) = \varphi(SH/H)$.

But $S \cap H = E$ so that by 5.2

$$\varphi(SH/H) = \varphi(S).H/H.$$

Therefore

$$\varphi(G) \leq \varphi(S).H \qquad \text{.... (A)}$$

On the other hand,

$$\varphi(G) = T \times \varphi(H)$$

by 8.2(c).

But G is soluble, so that by a theorem due to P. Hall [9], T lies in some conjugate S^x of S, and since T is normal in G, we have

$$T \leq S.$$

Therefore

$$\varphi(G) \leq S.\varphi(H) \qquad \text{.... (B)}$$

Comparing (A) with (B), we obtain

$$\varphi(G) \leq \varphi(S).\varphi(H).$$

Therefore

$$\varphi(G) = \varphi(G) \cap (\varphi(S).\varphi(H)) = (\varphi(G) \cap \varphi(S)).\varphi(H).$$

Since $\varphi(G)$ is nilpotent, $\varphi(H)$ has a unique complement so that

$$\varphi(G) \cap \varphi(S) = T, \text{ and } \varphi(G) = (\varphi(G) \cap \varphi(S)) \times \varphi(H).$$

CHAPTER 1X

FRATTINI SUBGROUPS OF L1-GROUPS

9.1 p-SERIES AND L1-GROUPS

The p-series of a finite group G is an ascending series

$$E \leq G_0 \leq G_1 \leq \cdots$$

of normal subgroups which are defined as follows.

G_0 is the largest normal p'-subgroup of G

G_1/G_0 is the largest normal p-subgroup of G/G_0

G_2/G_1 is the largest normal p'-subgroup of G/G_1, and so on.

If, eventually, $G_K = G$ for some $K \geq 0$, G is said to be p-soluble, and the number of p-factors in its p-series is called the p-length of G.

A finite group having p-length 1 for every prime p dividing its order will be called an L1-group.

9.2 THE MAIN THEOREM

The following theorem describes the Frattini subgroup of an L1-group.

9.2.1 Let G be an L1-group, and for each prime p let G_p be a Sylow p-subgroup of G, and put

$$D_p = \bigcap_{g \in G} \varphi(G_p)^g$$

Then $\varphi(G) = \prod_p D_p$.

B. Huppert [13] has proved 9.2.1 for finite groups having a nilpotent commutator subgroup.

<u>Proof</u> We first prove the following lemma.

<u>Lemma</u> Let G be a finite group of p-length 1, and let $E \leq K < H \leq G$ be its p-series. Then

$$\varphi(G/K) = \varphi(H/K) .$$

For, H/K is the Sylow p-subgroup of G/K so that by 8.2,

$$\varphi(G/K) = \varphi(H/K) \times T$$

where T is a p'-subgroup. But G/K has no non-trivial normal p'-groups from the way in which the p-series of G is defined. Therefore $T = E$.

Returning to the proof of 9.2.1, each D_p is normal in G and omissible in G_p so that by 3.1.1, D_p is omissible in G itself. Consequently,

$$\prod_p D_p \leq \varphi(G).$$

Let φ_p be the Sylow p-subgroup of $\varphi(G)$.
We shall complete the proof by showing that $\varphi_p \leq D_p$.
We have

$$K.\varphi_p/K \leq K.\varphi(G)/K \leq \varphi(G/K).$$

But

$$\varphi(G/K) = \varphi(H/K)$$

by the lemma.

Also $H = K.G_p$ where $K \cap G_p = E$, so that by 5.2

$$\varphi(H/K) = \varphi(K.G_p/K) = K.\varphi(G_p)/K.$$

Therefore $\varphi_p \leq K.\varphi(G_p)$, and $K.\varphi(G_p)$ is normal in G. Since $\varphi(G_p)$ is a Sylow p-subgroup of $K.\varphi(G_p)$, and φ_p is normal in G, we obtain

$$\varphi_p \leq \bigcap_{g \in G} \varphi(G_p)^g = D_p$$

thereby completing the proof of 9.2.1.

9.3 A COUNTER-EXAMPLE

We conclude this chapter with an example, due to L.G. Kovacs, of a finite group having 3-length 2 and p-length ≤ 1 $(p \neq 3)$ for which

$$\bigcap_p D_p < \varphi(G) .$$

We first prove a simple lemma.

<u>Lemma</u> Let A be an abelian normal subgroup of a group G, and let N be a minimal normal subgroup of G which lies in A. Suppose that N has no complement in A which is normal in G. Then $N \leq \varphi(G)$.

<u>Proof</u> Suppose not, and let M be a maximal subgroup of G which does not contain N. Then $M \cap N < N$, and $MN = G$. $M \cap N$ is normal in M, and normal in N because N is abelian. Consequently, $M \cap N$ is normal in G, and therefore $M \cap N = E$ by the minimal normality of N.

We have $A = (A \cap M)N$, $(A \cap M) \cap N = E$. But $A \cap M$ is normal in G because it is normal in M and centralized by N. This contradicts the assumption that N has no G-normal complement in A, and proves the lemma.

We now construct a group G as follows.

Let A be an elementary abelian 3-group of order 3^4 with generators a,b,c and d.

Let Q be the quaternion group of order 8:

$$Q = gp.\left\{s,t : s^4 = e,\ s^2 = t^2,\ s^t = s^{-1}\right\} .$$

Q admits an automorphism σ of order 3, where

$$s^\sigma = t, \quad t^\sigma = st.$$

Put $\Sigma = gp.\left\{\sigma\right\}$, and let $Q\Sigma$ be the holomorph of Q by Σ. It is a routine matter to check that A admits $Q\Sigma$ as a group of automorphisms, where

$$a^s = b^{-1}, \quad b^s = a, \quad c^s = d^{-1}, \quad d^s = c$$
$$a^\sigma = a, \quad b^\sigma = ab, \quad c^\sigma = ac, \quad d^\sigma = abcd .$$

We define G to be the holomorph of A by $Q\Sigma$.

Let $Z = \text{sgp.}\{s^2\}$; Z is the centre and also the commutator subgroup of Q.

Let $N = \text{sgp.}\{a,b\}$.

It is not hard (merely tedious!) to show that the only normal subgroups of G are E, N, A, AZ, AQ and G.

We establish a number of properties of G.

(a) G has 3-length 2 and p-length ≤ 1 if $p \neq 3$.

For $E < N < A < AZ < AQ < G$ are the only normal subgroups of G, and an examination of the factors proves (a).

(b) $\varphi(G) = N$

$\varphi(G)$ is normal and omissible in G, and therefore must be E or N. But N has no G-normal complement in A, so that by the previous lemma, $N \leq \varphi(G)$. This proves (b).

(c) $\bigcap_p D_p = E$, where $D_p = \bigcap_{g \in G} \varphi(G_p)^g$.

Only D_2 and D_3 are relevant.

D_2 must be E because G has no non-trivial normal 2-subgroups. $A\Sigma$ is a Sylow 3-subgroup of G, so we may take $G_3 = A\Sigma$.

$\delta(G_3)$ is generated by the commutators

$[a,\sigma] = e, \quad [b,\sigma] = [c,\sigma] = a, \quad [d,\sigma] = abc$

and their conjugates in G_3.

Therefore $\delta(G_3) = \text{sgp.}\{a, bc\}$.

The factor group $G_3/\delta(G_3)$ is elementary abelian, so that $\beta_3(G_3) \leq \delta(G_3)$.

Therefore $\varphi(G_3) = \beta_3(G_3).\delta(G_3) = \text{sgp.}\{a, bc\}$.

Since this subgroup does not contain the only minimal G-normal subgroup (monolith) N, we must have

$\bigcap_{g \in G} \varphi(G_3)^g = E$, i.e., $D_3 = E$. This proves (c), and

thereby completes the example.

CHAPTER X

THE CLASS OF FRATTINI SUBGROUPS

In this chapter, we consider the class of all Frattini subgroups, and begin with a theorem due to V. Dlab [2].

10.1 If H is an abelian group, then there exists an abelian extension K of H with the following properties:

(a) $H = \varphi(K)$

(b) If G is any abelian extension of H such that $H \leq \varphi(G)$, then the identity map of H can be extended to a monomorphism from K to G.

<u>Proof</u> We recall that H may be embedded in a minimal divisible group D with the property that every non-trivial subgroup of D meets H non-trivially.

Let K/H be the socle of D/H; K/H is the union of all the prime-order subgroups of D/H. We show that K has the required properties.

<u>To prove</u> (a), observe that $\varphi(K/H) = E$, so that $\varphi(K) \leq H$. Conversely, let x be any element of H and let p be any prime. Using additive notation, $x = py$ for some element y in D since D is divisible. But y must lie in K because py lies in H. Therefore x is divisible by p within K, giving $H \leq \varphi(K)$.

<u>To prove (b)</u>, let G be an abelian extension of H such that $H \leq \varphi(G)$. We consider pairs (L, θ) where L is a subgroup of K containing H, and θ is a homomorphism from L to G which maps H identically.

These pairs may be partially ordered by setting $(L_1,\theta_1) \leq (L_2,\theta_2)$ whenever L_2 contains L_1 and θ_2 agrees with θ_1 on L_1. Zorn's lemma may clearly be applied to obtain a maximal pair (M,ψ).

We show that $M = K$. For if not, then since K/H is a direct product of elementary abelian groups, there exists a prime p and an element x of K such that $\text{sgp.}\{x,H\}/H$ has order p and meets M/H trivially.

Let $M_1 = \text{sgp.}\{x,M\}$. Then $M_1 > M$, and each element v of M_1 is uniquely expressible as

$$v = tx + u$$

where $0 \leq t \leq p-1$ and u lies in M.

Since px belongs to H which in turn is contained in $\varphi(G)$, we have

$$px = pg$$

for some element g of G.

The homomorphism $\psi : M \longrightarrow G$ may now be extended to $\psi_1 : M_1 \longrightarrow G$ by setting

$$(tx + u)\psi_1 = tg + u\psi,$$

in contradiction to the maximal choice of (M,ψ).

It remains to show that ψ is a monomorphism. Let N be the kernel of ψ. Then $N \cap H = E$ because ψ maps H identically. But then $N = E$ because N is contained in the minimal divisible extension D of H.

10.1 is interesting because it shows that every abelian group occurs as a Frattini subgroup. In the opposite direction, the following theorem narrows the class of Frattini subgroups considerably.

10.2 If G is any group such that $\varphi(G)$ is finite, then $\varphi(G)$ is nilpotent. This is a special case of the following theorem.

10.3 If H is an omissible normal finite subgroup of a group G, then H is nilpotent.

<u>Proof</u> We prove that H is nilpotent by showing that every Sylow subgroup of H is normal in H.

Let P be a Sylow subgroup of H, and let N be the normalizer of P in G. If g is any element of G, then P^g and P are conjugate within H,

i.e.,
$$P^g = P^h$$
for some element h of H. But then $gh^{-1} \in N$, so that $g \in NH$. Therefore $G = NH$. Since H is omissible in G, this reduces to

$$G = N,$$

showing that P is normal in G. P is therefore normal in H, as required.

The first result in this direction was obtained by G. Frattini who showed that $\varphi(G)$ is nilpotent if G is finite.

The previous theorem poses the question: can every finite nilpotent group H be extended to a group G in which it is omissible?

The following example shows that this is not always possible. Take H to be the dihedral group of order 8, i.e.,

$$H = \mathrm{gp.}\left\{a,b : a^4 = b^2 = e,\ a^b = a^{-1}\right\}$$

The subgroup $A = \mathrm{sgp.}\{a\}$ is characteristic in H (being the only cyclic subgroup of order 4).

Now suppose that H is a normal subgroup of a group G. Then A is normal in G, and therefore its centralizer C in G is also normal in G. Now G/C is isomorphic to a subgroup of the automorphism group of A so that G/C is either trivial or cyclic of order 2. We cannot have $C = G$ because H does not centralise A. Therefore C is a maximal subgroup of G not containing H. Consequently, H cannot lie in $\varphi(G)$ and so cannot be omissible in G.

CHAPTER X1

A THEOREM OF M.F. NEWMAN

Let D be a normal subgroup of a group H. If H is nilpotent, then so is H/D. M.F. Newman's theorem describes certain conditions under which it is possible to prove that H is nilpotent, given that H/D is nilpotent. The precise statement of the theorem is:

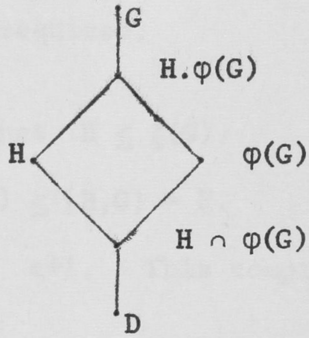
11.1 If D and H are normal subgroups of a group G such that

(i) $D \leq H \cap \varphi(G)$,

(ii) H/D is nilpotent,

(iii) D is polycyclic-by-finite,

then H is nilpotent.



Before proving 11.1., we give three results which will be needed during the proof of 11.1.

11.2 If G is a p-group with a finite normal subgroup H of order p^n, and if G/H is nilpotent of class c, then G is nilpotent of class at most $c + n$.

<u>Proof</u> Suppose that $n > 0$, the theorem being trivial if $n = 0$.

Firstly, we show that $H \cap \zeta(G) > E$.

H splits into conjugacy classes $C_1 = \{e\}, C_2, \ldots, C_k$ in G.

If C_i contains m_i elements, then $m_1 = 1$ and m_i $(i \geq 2)$ is either 1 or a positive power of p.

The equation

$$m_1 + m_2 + \ldots + m_k = p^n$$

now shows that besides m_1, at least $p-1$ other m's are equal to 1.

Therefore H contains a central element other than e. Put $K = H \cap \zeta(G)$. Then K is normal in G and has order p^s where $1 \leq s \leq n$. Therefore H/K has order p^{n-s} so that by induction, G/K is nilpotent of class at most $c + (n-s)$.

If $s < n$, then a second induction shows that G is nilpotent of class at most $c + (n-s) + s = c+n$, as required.

If, however, $s = n$, then $K = H$ so that $H \leq \zeta(G)$. In this case, $\gamma_{c+1}(G) \leq H$, and so $\gamma_{c+2}(G) \leq [H,G] = E$. Therefore G is nilpotent of class at most $c+1$. This completes the proof.

11.3 If G is a finitely generated group, then every normal subgroup of finite index contains a characteristic subgroup of finite index.

<u>Proof</u> (M. Hall [8]). Let N be a normal subgroup whose index m is finite. If M is <u>any</u> normal subgroup of index m, then the composition $G \longrightarrow G/M \longrightarrow S_m$ defines a homomorphism of G into the symmetric group of degree m whose kernel is precisely M. Since G is finitely generated and S_m is finite, there are only finitely many homomorphisms of G into S_m. Consequently G has only finitely many normal subgroups of index m.

If D is their intersection, then D is a characteristic subgroup of finite index, and $D \leq N$.

11.4 If H is a normal subgroup of a group G, and if $G/\delta(H)$ and H are nilpotent, then G is nilpotent. (P. Hall [10]).

Proof of 11.1 The proof we give is due to M.F. Newman, and is divided into several parts. We begin by proving the theorem under certain additional conditions, and gradually relax these conditions until we obtain the theorem in its most general form.

Part 1. Assume that H is finite.

Let P/D be the Sylow p-subgroup of H/D. P/D is characteristic in H/D, therefore normal in G/D. Therefore P is normal in G.

Let S be a Sylow p-subgroup of P. S is also a Sylow p-subgroup of H since the index of P in H is prime to p. If N is the normaliser of S in G, and if $x \in G$, then $S^g = S^x$ for some $x \in P$. But then $gx^{-1} \in N$, so that $G = NP$. Also, $P = DS$ since P/D is a p-group and S is a Sylow p-subgroup of P. Consequently,

$$\begin{aligned} G = NP &= NDS \\ &= ND \quad \text{(because } S \leq N\text{)} \\ &= N \quad \text{(because } D\text{, being finite, is omissible in } G\text{).} \end{aligned}$$

Hence $S \trianglelefteq G$, $S \trianglelefteq H$, therefore H is nilpotent.

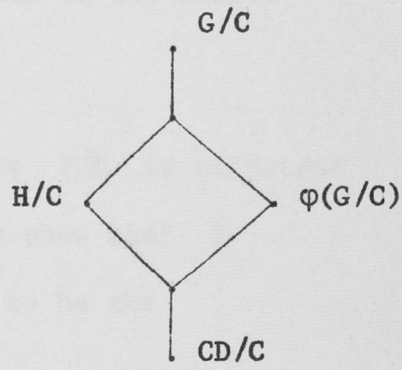
Part 2. Assume that D is finite and minimal-normal in G.

Let C be the centraliser of D in H. C is normal in G, for if $c \in C$, $d \in D$, $g \in G$, then $[c^g,d] = [c,d^{g^{-1}}]^g = e^g = e$, so that $c^g \in C$.

Therefore $C \cap D$ is normal in G, and by the minimal property of D, either $C \cap D = E$ or $C \cap D = D$. We therefore distinguish

Case (i) $C \cap D = E$.

In the group G/C we have the lattice indicated, in which H/C is finite (as a group of automorphisms of the finite group D). Part 1 therefore applies to show that H/C is nilpotent. But $C \cap D = E$, so that H is isomorphic to a subgroup of the direct product of the nilpotent groups H/C and H/D. Consequently, H is nilpotent.



Case (ii) $C \cap D = D$, i.e., $D \leq C$.

In this case, D is a finite abelian group, and by the minimal property it must be an elementary abelian p-group for some prime p. Suppose that D has order p^n, and H/D is nilpotent of class c.

Then C is nilpotent of class $\leq c+1$ because

$$\gamma_{c+2}(C) = [\gamma_{c+1}(C),C] \leq [\gamma_{c+1}(H),C] \leq [D,C] = E.$$

We now consider three cases, (a), (b) and (c), each one less restrictive on C.

(a) Suppose that C is an elementary abelian p-group. We shall show that H is the direct product of its Sylow q-subgroups H_q, where

H_q is nilpotent of class $\leq c$ if $q \neq p$, and

H_p is nilpotent of class $\leq c+n$.

It will then follow that H is nilpotent (of class $\leq c+n$).

Firstly, H is periodic because H/C is finite and C is a p-group. Hence H/D is a periodic nilpotent group, and is the direct product of its Sylow subgroups.

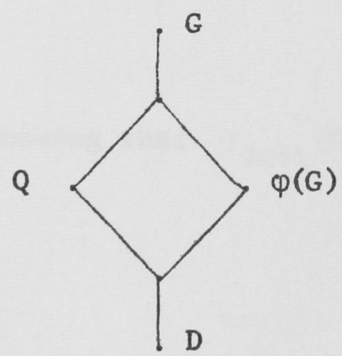
Let P/D be the Sylow p-subgroup of H/D. Since P/D is nilpotent of class $\leq c$ and D has order p^n, 11.2 applies to show that P is nilpotent of class $\leq c+n$. But P is easily seen to be the unique Sylow p-subgroup of H, i.e., $P = H_p$.

For $q \neq p$, let Q/D be the Sylow q-subgroup of H/D. Then Q/D is mapped faithfully in H/C by the homomorphism $\theta : Q/D \longrightarrow H/C$ defined by

$$(xD)\theta = xC \qquad (x \in Q).$$

This shows that Q/D is finite and nilpotent of class $\leq c$. Since D is finite, Q is also finite.

The next step is to show that Q is nilpotent. In G we have the lattice to the right where Q is finite and Q/D is nilpotent. Part 1 now shows that Q is nilpotent. Let T be the Sylow q-subgroup of Q. Then T is seen to be the (unique) Sylow q-subgroup of H. i.e., $T = H_q$.

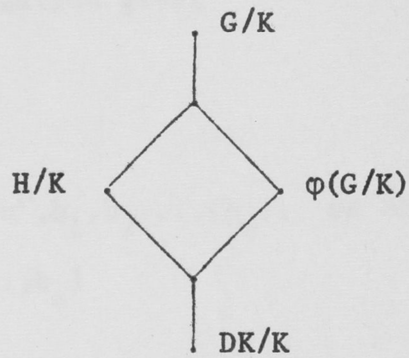


Since Q/D has class $\leq c$, we have $\gamma_{c+1}(T) \leq \gamma_{c+1}(Q) \leq D$. But T is a q-group, and D a p-group, so that $\gamma_{c+1}(T) = E$. Therefore T is nilpotent of class $\leq c$. This completes the proof of (a).

(b) Suppose that C is abelian. Let $K = \beta_p(C)$. Then K is normal in G so that by the minimal property of D, either $D \cap K = E$ or $D \leq K$. We consider these possibilities separately.

<u>$D \cap K = E$</u>

In G/K we have the lattice to the right. The centraliser of DK/K is precisely C/K which is an elementary abelian p-group. The remaining conditions of (a) are easily checked, so that by (a), H/K is nilpotent. Since $D \cap K = E$ H is isomorphic to a subgroup of the direct product of the nilpotent groups H/D and H/K. Therefore H is nilpotent.



<u>$D \leq K$</u>

We shall prove that H is nilpotent by showing that $\gamma_{2c+1}(H) = E$. Since H/D has class c, we have

$$\gamma_{c+1}(H) \leq D \leq K$$

Let $g = [h_1', h_2', \ldots, h_{c+1}', h_1, h_2, \ldots, h_c]$ be any commutator of length $2c+1$ whose entries belong to H. We must show that $g = e$. The commutator $x = [h_1', h_2', \ldots, h_{c+1}']$ belongs to K, so that $x = u^p$ for some element $u \in C$. Therefore $g = [u^p, h_1, h_2, \ldots, h_c]$.

The next step is to show that the last commutator is equal to $[u, h_1, h_2, \ldots, h_c]^p$. If $v \in C$, $h \in H$ and $s \geq 1$, then

$$[v^{s+1}, h] = [v^s, h]^v [v, h]$$

$$= [v^s, h][v, h] \quad \text{(since } C \text{ is abelian)}$$

An obvious induction applied to this equation gives

$$[v^s, h] = [v, h]^s$$

Applying this c times to the commutator $[u^p, h_1, h_2, \ldots, h_c]$, we obtain in turn $g = [u^p, h_1, \ldots, h_c] = [[u, h_1]^p, h_2, \ldots, h_c]$

$$= [[u, h_1, h_2]^p, h_3, \ldots, h_c] = \text{etc.}$$

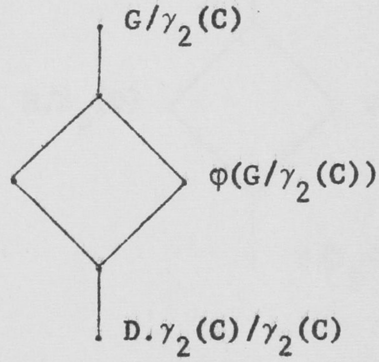
$$= [u, h_1, h_2, \ldots, h_c]^p.$$

But $[u, h_1, \ldots, h_c] \in \gamma_{c+1} \leq D$, and D is an elementary abelian p-group, so that $g = [u, h_1, \ldots, h_c]^p = e$. This completes the proof of (b).

(c) No restriction on C.

In any case, C is nilpotent so that there exists an integer $r \geq 1$ such that $\gamma_r(C)$ contains D but $\gamma_{r+1}(C)$ does not. By the minimal property of D, we must have $D \cap \gamma_{r+1}(C) = E$.

If $r = 1$, then in $G/\gamma_2(C)$ we have the lattice to the right.

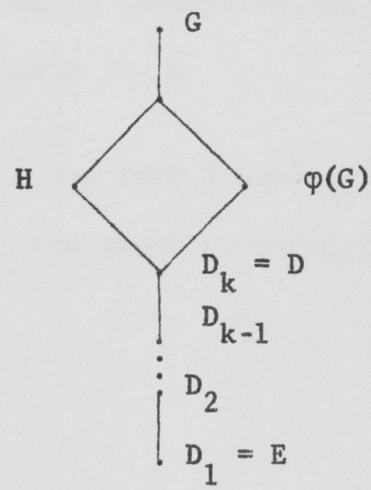


The centraliser of $D.\gamma_2(C)/\gamma_2(C)$ in $H/\gamma_2(C)$ is seen to be the abelian group $C/\gamma_2(C)$.

The remaining conditions of (b) are easily seen to be satisfied, so that by (b), $H/\gamma_2(C)$ is nilpotent. Since $D \cap \gamma_2(C) = E$, H is isomorphic to a subgroup of the direct product of the nilpotent groups H/D and $H/\gamma_2(C)$, so that H is nilpotent.

If $r > 1$, then $D \leq \gamma_2(C)$ so that $H/\gamma_2(C)$ is isomorphic to a factor group of the nilpotent group H/D. So in this case, $H/\gamma_2(C)$ and C are both nilpotent. The nilpotency of H now follows from 11.4. This proves (c), and thereby completes the proof of Part 2.

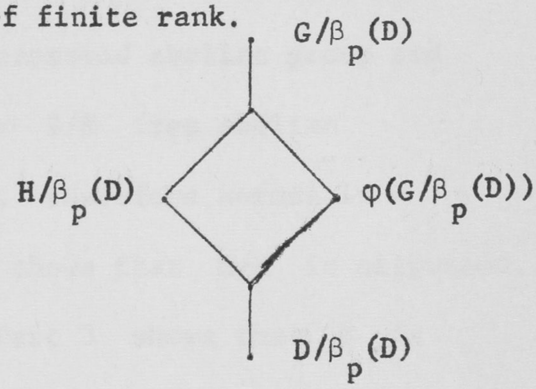
<u>Part 3</u>. Suppose that D is finite.



Let $E = D_1 < D_2 < \ldots < D_k = D$ be a series of subgroups, all normal in G, such that D_{i+1}/D_i is minimal-normal in G/D_i.

By applying Part 2., k-1 times, we obtain in turn that the groups $H/D_{k-1}, H/D_{k-2}, \ldots, H/D_1 = H$ are nilpotent.

<u>Part 4</u>. Suppose that D is free abelian of finite rank.



Let D have rank k, and let p be any prime. In $G/\beta_p(D)$ we have the lattice indicated, in which $D/\beta_p(D)$ has finite order p^k. Part 3 now applies to show that $H/\beta_p(D)$ is nilpotent.

Since $\gamma_{c+1}(H) \leq D$ and $[D : \beta_p(D)] = p^k$, we must have

$$\gamma_{c+1+k}(H) \leq \beta_p(D).$$

This is true for every prime p, therefore

$$\gamma_{c+k+1}(H) \leq \bigcap_p \beta_p(D) = E.$$

Therefore H is nilpotent (of class $\leq c+k$).

<u>Part 5</u>. (the last!) Suppose that D is polycyclic-by-finite.

D contains a polyclic normal subgroup B such that D/B is finite. Every subgroup of a polycyclic group is polycyclic, so that by 11.3, we may assume that B is characteristic in D.

In G/B, the subgroup D/B is finite, and Part 3 applies to show that H/B is nilpotent. We shall complete the proof by using induction on the soluble length $\ell(B)$ of B.

If $\ell(B) = 1$, then B is a finitely generated abelian group and therefore contains a finite subgroup K with B/K free abelian of finite rank. K is characteristic in B, therefore normal in G, and Part 4, applied in the group G/K, shows that H/K is nilpotent. Since K is finite, a final application of Part 3 shows that H is nilpotent.

If $\ell(B) > 1$, then $B/\delta(B)$ is a finitely generated abelian group, and the previous case implies that $H/\delta(B)$ is nilpotent. But $\delta(B)$ has smaller soluble length than B, so that by induction, H is nilpotent. This completes the proof of 11.1.

11.5 The behaviour of the Fitting radical under homomorphisms.

We recall that the Fitting radical $\mathfrak{F}(G)$ of a group G is the subgroup generated by all the nilpotent normal subgroups of G. If θ is a homomorphism of G into some group $\bar{G}$, then it is obvious from the definition of $\mathfrak{F}$ that

$$(\mathfrak{F}(G))\theta \leq \mathfrak{F}(G\theta).$$

Using Newman's theorem, we can show that if the kernel of θ is a finitely generated nilpotent subgroup of $\varphi(G)$, then

$$(\mathfrak{F}(G))\theta = \mathfrak{F}(G\theta)$$

We now state and prove this formally.

6. If D is a normal finitely generated nilpotent subgroup of a group G, and if $D \leq \varphi(G)$, then

$$\mathfrak{F}(G)/D = \mathfrak{F}(G/D)$$

<u>Proof</u> It is only necessary to prove the inclusion $\mathfrak{F}(G/D) \leq \mathfrak{F}(G)/D$ since the reverse inclusion is obvious.

Let H/D be a normal nilpotent subgroup of G/D. Since a finitely generated nilpotent group is polycyclic; H and D are seen to satisfy the conditions of 11.1., so that by 11.1, H is nilpotent. Therefore $H \leq \mathfrak{F}(G)$, and $H/D \leq \mathfrak{F}(G)/D$. Since H/D is any normal nilpotent subgroup of G/D, we obtain $\mathfrak{F}(G/D) \leq \mathfrak{F}(G)/D$ as required.

CHAPTER X11

A THEOREM ON DIRECT PRODUCTS

If A and B are groups, it is still an open question whether

$$\varphi(A \times B) = \varphi(A) \times \varphi(B)$$

for every A and B. This question is closely related to another open question; namely, are there any simple groups without maximal subbroups? The precise connection between these questions is given by the following theorem due to V. Dlab and V. Kořínek [3].

12.1 There exist groups A and B such that

$$\varphi(A \times B) \neq \varphi(A) \times \varphi(B)$$

if and only if there is a simple group without maximal subgroups.

The proof we give is based on a proof given by Dr. M.F. Newman in his course on Frattini subgroups. We begin with some definitions.

<u>Definition</u>. If G is any group, let $\varphi_N(G)$ be the intersection of all the maximal-normal subgroups of G, or else G if G has none.

<u>Definition</u>. By an <u>obvious</u> maximal subgroup of a direct product $A \times B$ of groups A,B is meant a maximal subgroup of the form $M_A \times B$ or $A \times M_B$ where M_A, M_B are maximal subgroups of A,B respectively.

<u>Definition</u>. A maximal subgroup of $A \times B$ which is not of the above forms will be called <u>non-obvious</u>.

Regarding obvious maximal subgroups of $A \times B$, the intersection of all the subgroups $M_A \times B$ is clearly $\varphi(A) \times B$, and of all the subgroups $A \times M_B$ is $A \times \varphi(B)$. Therefore

$$\varphi(A \times B) \leq (\varphi(A) \times B) \cap (A \times \varphi(B)) = \varphi(A) \times \varphi(B).$$

Regarding non-obvious maximal subgroups of $A \times B$, we have

12.1.1 If M is a non-obvious maximal subgroup of $A \times B$, then

(a) M has full projection on A and on B

(b) $M \cap A$ is maximal-normal in A

(and similarly for $M \cap B$ in B)

(c) $\varphi_N(A) \times \varphi_N(B)$ lies in every non-obvious maximal subgroup of $A \times B$.

Proof of (a).

Let M_A be the projection of M into A. If M_A were properly contained in A, we should have

$$M \leq M_A \times B < A \times B$$

so that $M = M_A \times B$. But then M would be obvious.

Proof of (b).

To show that $M \cap A$ is normal in A, let $a \in A$. By (a), B contains an element b such that $ab \in M$. But then $(M \cap A)^a = (M \cap A)^{ab} \leq M \cap A$.

To show that $M \cap A < A$, if we had $M \cap A = A$, we should have $A \leq M$ and therefore $B \leq AM = M$. But then $M = A \times B$.

To show that $M \cap A$ is maximal-normal in A, let y be an element of A outside $M \cap A$. We must show that the A-normal closure of y and $M \cap A$ is A. Let $a \in A$. Since $y \notin M$, we have

$$\text{sgp.}\{y,M\} = A \times B$$

so that

$$a = y^{\ell_1} m_1 \ldots y^{\ell_r} m_r y^{\ell_{r+1}} \qquad (m_i \in M,\ \ell_i \text{ integers})$$

But then

$$a = m_1 \ldots m_r \, (y^{\ell_1})^{m_1 \ldots m_r} (y^{\ell_2})^{m_2 \ldots m_r} \ldots y^{\ell_{r+1}}$$

where $m_1 \ldots m_r$ lies in $M \cap A$ since all the other factors lie in A. Thus a lies in the normal closure of y and $M \cap A$.

Proof of (c).

Let M be a non-obvious maximal subgroup of $A \times B$. Then by (b),

$$\varphi_N(A) \leq M \cap A \quad \text{and} \quad \varphi_N(B) \leq M \cap B$$

so that

$$\varphi_N(A) \times \varphi_N(B) \leq M .$$

We need one more result before proving 12.1.

12.1.2 If all simple factor groups of a group G have maximal subgroups, then

$$\varphi(G) \leq \varphi_N(G).$$

<u>Proof</u> Let H be maximal-normal in G. Then G/H is simple and therefore has a maximal subgroup. But then

$$\varphi(G/H) < G/H$$

so that

$$\varphi(G/H) = E.$$

Therefore $\varphi(G) \leq H$.

<u>Proof of 12.1</u>

Firstly, suppose that every simple group has maximal subgroups. Then 12.1.2 applies to give

$$\varphi(A) \leq \varphi_N(A) \quad \text{and} \quad \varphi(B) \leq \varphi_N(B).$$

Therefore

$$\varphi(A) \times \varphi(B) \leq \varphi_N(A) \times \varphi_N(B).$$

This inequality and 12.1.1 (c) show that $\varphi(A) \times \varphi(B)$ is contained in every non-obvious maximal subgroup of $A \times B$. On the other hand, we have shown that $\varphi(A) \times \varphi(B)$ is the intersection of all the obvious maximal subgroups of $A \times B$. Hence

$$\varphi(A) \times \varphi(B) \leq \varphi(A \times B).$$

The reverse inclusion has already been proved, giving
$\varphi(A \times B) = \varphi(A) \times \varphi(B)$.

Secondly, suppose that A is a simple group without maximal subgroups. We shall prove that $\varphi(A \times A) < \varphi(A) \times \varphi(A)$.

Firstly, $\varphi(A) = A$ since A has no maximal subgroups. Hence

$$\varphi(A) \times \varphi(A) = A \times A.$$

Write the elements of $A \times A$ as (a_1,a_2), and let D be the set of all diagonal elements (a,a). D is obviously a subgroup of $A \times A$. D is even maximal, for suppose that $(b,c) \notin D$, and let

$$H = \text{sgp.}\{D,(b,c)\}$$

Then

$$H = \text{sgp.}\{D,(e,u)\}$$

where $u = b^{-1}c$, since $(b,c) = (b,b)(e,b^{-1}c)$.

If $a \in A$, then

$$(e,u^a) = (e,u)^{(a,a)} \in H.$$

But the conjugates u^a of u generate A because $u \neq e$ and A is simple.

Therefore

$$E \times A \leq H.$$

Similarly,

$$A \times E \leq H,$$

giving

$$H = A \times A.$$

Finally,

$$\varphi(A \times A) \leq D < A \times A,$$

so that

$$\varphi(A \times A) < \varphi(A) \times \varphi(A).$$

Note. An examination of the proof of 12.1 shows that if A and B are groups all of whose simple factor groups have maximal subgroups, then

$$\varphi(A \times B) = \varphi(A) \times \varphi(B).$$

G.A. Miller [19] first proved this relation for finite groups A and B.

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